**Comparing Marxen BusyBeaver with Ackermann function**

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## Introduction

Remember definition of Ackerman function monotone in both variables:

$$A(x,0)=2$$

$$A(0,y)=y+2$$

$$A(x+1,y+1)=A(x,A(x+1,y))$$

One variable variant read from diagonal $A(x)=A(x,x)$. Remember that $A(1,y)=A(0,A(1,y-1))=A(1,y-1)+2=A(1,0)+2y=2y+2>2y$, $A(2,y)=A(1,A(2,y-1))>2A(2,y-1)>2^yA(2,0)=2^y\cdot 2>2^y$, $A(3,y)=A(2,A(3,y-1))>2^{A(3,y-1)}>(2\uparrow y)(A(3,0))>2\uparrow y$ where $(2\uparrow y)(x)$ denotes $f^{(y)}(x)$ for $f(x)=2^x$ and $2\uparrow y$ is shortcut for $f^{(y)}(1)$. $A(4,y)=A(3,A(4,y-1))>2\uparrow A(4,y-1)>(2\Uparrow y)(A(4,0))>(2\Uparrow y)$ where $(2\Uparrow y)(x)$ denotes $f^{(y)}(x)$ for $f(x)=2\uparrow x$ and $2\Uparrow y$ is shortcut for $f^{(y)}(1)$. $A(6)$ is bigger than any number connected with the Universe.

Busy Beaver $S(n)$ function is the maximal number of steps deterministic Turing machine with $n$ states working on $\{0,1\}^\star$ tape initialised with $0^\star$ can process and stop. Our proof works also for function defined as the maximal number of ones deterministic Turing machine with $n$ states working on $\{0,1\}^\star$ tape initialised with $0^\star$ can left on tape and stop.

## Sketch of proof that $S(61)>A(A(6))$

We will create TM with at most 61 states working on 0,1 tape initialy filled with $0^\star$ which will generate $0^\star 1^x 0^\star$ where $x>A(A(6))$. Obviously the running time would be higher.

Construction of TM will be devided into several parts. Original plan for first part was to prepare configuration $0^\star 10S{\rightarrow}01^7 0^\star$. Then computation of $A^{(i+1)}(w)$ would be invoked from state $0^\star 1^i 0S{\rightarrow}01^{w+1}0^\star$. Computation of $A(w)$ transforms $|0S{\rightarrow}01^{w+1}0^\star$ to $|0^2b{\rightarrow}1^{w+1}01^{w+1}0^\star$. Third part should transform $|0^2b{\rightarrow}1^{x+1}01^{y+1}0^\star$ to $|{\leftarrow}z0^21^{A(x,y)+1}0^\star$. Fourth part transforms $0^\star 1^i{\leftarrow}z0^21^{x+1}0^\star$ to $0^\star 1^{i-1}0S{\rightarrow}01^{x+1}0^\star$ and stops for $|0{\leftarrow}z0^21^{x+1}0^\star$.

Alltogether $0^\star I{\rightarrow}0^\star \Rightarrow 0^\star 10S{\rightarrow}01^{6+1}0^\star \Rightarrow 0^\star 10^2b{\rightarrow}1^{6+1}01^{6+1}0^\star \Rightarrow 0^\star 1{\leftarrow}z0^21^{A(6,6)+1}0^\star \Rightarrow 0^\star 0S{\rightarrow}01^{A(6)+1}0^\star \Rightarrow 0^\star b{\rightarrow}1^{A(6)+1}01^{A(6)+1}0^\star \Rightarrow 0^\star{\leftarrow}z0^21^{A(A(6),A(6))+1}0^\star = 0^\star{\leftarrow}z0^21^{A(A(6))+1}0^\star$ where it stops.

## Improvement to $S(35)>A(A(10))$

Finally I have reused some parts of program which allowed to reduce states count to 35. The first part using best Busy Beaver for 4 states prepares $0^\star 10S{\rightarrow}11^{10+1}0^\star$ instead. Actually I have resigned on construction of $A^{(i+1)}(w)$. I used $B^{(i+1)}(w)$ instead, where $B(w)=A(w,2w+3)>A(w)$. I have replaced second part with transformation $|0S{\rightarrow}(0|1)1^{w+1}0^\star$ to $|0^2b{\rightarrow}1^{w+1}01^{2w+3+1}0^\star$. Moreover state pairs $(b,d)$ can be unioned as well as $(B,D)$ so there exists beaver with 35 states creating more than $A(A(6))$ ones (actually $B^{(2)}(10)>A(A(10))$).

See the commented transition function on the next page.

With 4 states Brady's initialisation we get $0^\star I0^\star \Rightarrow 0^\star 101^9{\leftarrow}X1^30^\star \Rightarrow 0^\star 10S{\rightarrow}11^{10+1}0^\star \Rightarrow 0^\star 10^2b{\rightarrow}1^{10+1}01^{23+1}0^\star \Rightarrow 0^\star 1{\leftarrow}z0^21^{A(10,23)+1}0^\star \Rightarrow 0^\star S{\rightarrow}01^{B(10)+1}0^\star \Rightarrow 0^\star b{\rightarrow}1^{B(10)+1}01^{2B(10)+4}0^\star \Rightarrow 0^\star{\leftarrow}z0^21^{A(B(10),2B(10)+3)+1}0^\star = 0^\star{\leftarrow}z0^21^{B(B(10))+1}0^\star$ being final configuration.

**Marxen BusyBeaver $S(35)>$ Ackermann $A(A(10))$.**

| **state** | **tape 0** | **tape 1** | **comment** | **configuration** |
|---|---|---|---|---|
| $a$ | $0\rightarrow b$ | $1\leftarrow a$ | backing on first argument of A(x,y) | $\vert 01^{x+1-k}\leftarrow a1^k01^{y+1}0^\star$ |
| | | | 1 variable case (end) | $\vert 001^{w+1-k}\leftarrow a1^k01^{2w+4}0^\star$ |
| $b=d$ | | $0\rightarrow c$ | stands on first 1 of first argument | $\vert 0b\rightarrow 1^{x+1}01^{y+1}0^\star$ |
| | | | when returned from r in nontrivial case | $\vert 01^x0b\rightarrow 1^{x+1}01^y0^\star$ |
| | | | when returned from a on the same level | $\vert 0b\rightarrow 1^x01^{A(x,y-1)+1}0^\star$ |
| | | | 1 variable case | $\vert 001^{w+1}0b\rightarrow 1^{2w+2}0^\star$ |
| | | | 1 variable case (end) | $\vert 00b\rightarrow 1^{w+1}01^{2w+4}0^\star$ |
| $c$ | $1\leftarrow d$ | $1\rightarrow f$ | was first argument 0? | $\vert 00c\rightarrow 1^x01^{y+1}0^\star$ |
| | | | 1 variable case | $\vert 001^{w+1}00c\rightarrow 1^{2w+1}0^\star$ |
| $d=b$ | $1\leftarrow e$ | | case A(0,y)=y+2 | $\vert 00\leftarrow d11^{y+1}0^\star$ |
| $e$ | $0\leftarrow y$ | | finalizing A(x,y) | $\vert 0\leftarrow e1^{A(x,y)+1}0^\star$ |
| $f$ | $0\rightarrow g$ | $1\rightarrow f$ | running to y | $\vert 001^{x-k}f\rightarrow 1^k01^{y+1}0^\star$ |
| | | | 1 variable case | $\vert 001^{w+1}001^{2w+1-k}f\rightarrow 1^k0^\star$ |
| $g$ | $(1\leftarrow D)$ | $0\rightarrow h$ | 0 finalizing for 1 variable else decrementing y | $\vert 001^x0g\rightarrow 1^{y+1}0^\star$ |
| | | | 1 variable case (end) | $\vert 001^{w+1}001^{2w+1}0g\rightarrow 0^\star$ |
| $h$ | $0\leftarrow j$ | $0\rightarrow n$ | was second argument 0? | $\vert 001^x00h\rightarrow 1^y0^\star$ |
| $j$ | $0\leftarrow j$ | $0\leftarrow k$ | running to x | $\vert 001^x0^\star\leftarrow j0^\star$ |
| $k$ | $0\rightarrow l$ | $0\leftarrow k$ | clearing x | $\vert 001^\star\leftarrow k0^\star$ |
| $l$ | $0\rightarrow m$ | | clearing x | $\vert 00l\rightarrow 0^\star$ |
| $m$ | $1\leftarrow c$ | | case A(x,0)=A(0,0)=2 | $\vert 000m\rightarrow 0^\star$ |
| $n$ | $1\leftarrow o$ | $1\rightarrow n$ | | $\vert 001^x0001^{y-1-k}n\rightarrow 1^k0^\star$ |
| $o$ | $0\leftarrow p$ | $1\leftarrow o$ | | $\vert 001^x0001^{y-k}\leftarrow o1^k0^\star$ |
| $p$ | $1\leftarrow p$ | $0\leftarrow q$ | | $\vert 001^x0^{2-k}\leftarrow p1^k01^y0^\star$ |
| $q$ | $1\rightarrow r$ | $1\leftarrow q$ | 2 variables case | $\vert 01^j01^{x-1-j-k}\leftarrow q1^k01^{2+j}01^y0^\star$ |
| | | | 1 variable case | $\vert 001^j01^{w-j-k}\leftarrow q1^k01^{2j+2}0^\star$ |
| $r$ | $0\rightarrow b$ | $0\rightarrow s$ | copy loop used both for 1 and 2 variables | $\vert 01^{j+1}r\rightarrow 1^{x-1-j}01^{2+j}01^y0^\star$ |
| | | | at the end of (j) copy loop | $\vert 01^xr\rightarrow 01^{x+1}01^y0^\star$ |
| | | | 1 variable case | $\vert 001^jr\rightarrow 1^{w-j+1}01^{2j}0^\star$ |
| $s$ | $0\rightarrow t$ | $1\rightarrow s$ | 2 variables case | $\vert 01^{j+1}01^{x-2-j-k}s\rightarrow 1^k01^{2+j}01^y0^\star$ |
| | | | 1 variable case | $\vert 001^j01^{w-j-k}s\rightarrow 1^k01^{2j}0^\star$ |
| $t$ | $1\rightarrow u$ | $1\rightarrow t$ | 2 variables case | $\vert 01^{j+1}01^{x-2-j}01^{2+j-k}t\rightarrow 1^k01^y0^\star$ |
| | | | 1 variable case | $\vert 001^j01^{w-j}01^{2j-k}t\rightarrow 1^k0^\star$ |
| $u$ | $(1\leftarrow x)$ | $0\rightarrow v$ | 0 for function with 1 variable, 1 for 2 variables | $\vert 01^{j+1}01^{x-2-j}01^{3+j}u\rightarrow 1^y0^\star$ |
| | | | 1 variable case | $\vert 001^j01^{w-j}01^{2j+1}u\rightarrow 0^\star$ |
| $v$ | $1\leftarrow w$ | $1\rightarrow v$ | | $\vert 01^{j+1}01^{x-2-j}01^{3+j}01^{y-1-k}v\rightarrow 1^k0^\star$ |
| $w$ | $0\leftarrow x$ | $1\leftarrow w$ | | $\vert 01^{j+1}01^{x-2-j}01^{3+j}01^{y-k}\leftarrow w1^k0^\star$ |
| $x$ | $0\leftarrow q$ | $1\leftarrow x$ | 2 variables case | $\vert 01^{j+1}01^{x-2-j}01^{3+j-k}\leftarrow x1^k01^y0^\star$ |
| | | | 1 variable case | $\vert 001^j01^{w-j}01^{2j+2-k}\leftarrow x1^k0^\star$ |
| $y$ | $0\leftarrow z$ | $1\leftarrow a$ | was this initial call? | $\vert 01^v\leftarrow y01^{A(x,y)+1}0^\star$ |
| $z$ | $0\rightarrow\star$ | $1\leftarrow A$ | was this last B(x) computation? | $0^\star1^{i-\ell}\leftarrow z001^{B^{(\ell)}(w)+1}0^\star$ |
| $A$ | $0\rightarrow B$ | $1\leftarrow A$ | move through i | $0^\star1^{i-\ell-k}\leftarrow A1^k001^{B^{(\ell)}(w)+1}0^\star$ |
| $B=D$ | | $0\rightarrow C$ | decrement i | $0^\star B\rightarrow 1^{i-\ell}001^{B^{(\ell)}(w)+1}0^\star$ |
| $C$ | $0\rightarrow S$ | $1\rightarrow C$ | | $0^\star1^{i-\ell-1-k}C\rightarrow 1^k001^{B^{(\ell)}(w)+1}0^\star$ |
| $S$ | $0\rightarrow r$ | $0\rightarrow r$ | | $0^\star1^{i-\ell-1}0S\rightarrow(0\vert 1)1^{B^{(\ell)}(w)+1}0^\star$ |
| $D=B$ | $1\leftarrow E$ | | 1 variable case (end) | $\vert 001^{w+1}001^{2w+1}0\leftarrow D10^\star$ |
| $E$ | $1\leftarrow F$ | $1\leftarrow E$ | 1 variable case (end) | $\vert 001^{w+1}001^{2w+3-k}\leftarrow E1^k0^\star$ |
| $F$ | $0\leftarrow a$ | | 1 variable case (end) | $\vert 001^{w+1}0\leftarrow F1^{2w+4}0^\star$ |
| $X$ | $0\rightarrow S$ | $1\leftarrow X$ | Brady's record 4 states beaver | $0^\star101^{12-k}\leftarrow X1^k0^\star$ |
| $I$ | $1\rightarrow J$ | $1\leftarrow J$ | Brady's A | |
| $J$ | $1\leftarrow I$ | $0\leftarrow K$ | Brady's B | |
| $K$ | $1\rightarrow X$ | $1\leftarrow L$ | Brady's C | |
| $L$ | $1\rightarrow L$ | $0\rightarrow I$ | Brady's D | |